Summary Notes on the Mathematical Theory of Finger Holes, as Applicable to Clarinets[1]

A. H. Benade, Case Institute of Technology, December 1960

An earlier paper on the properties of woodwinds (referred to here as paper A)[2] showed that, as a consequence of the use of finger holes, the only musically useful bores are members of the Bessel horn family in which the cross-sectional area *S(x)* of the bore varies as some single positive power of the distance x from the reed end. Various other considerations further limit the bores to two members of this family. The straight, cylindrical bore of the clarinet group is a representative of one of these remaining bores, ($\epsilon = 0$), while the conical bores used in oboes, saxophones, etc., belong to the remaining possibility, ($\epsilon = 2$). The discussion in paper A was based on the assumption that opening a series of side holes in the lower part of the bore is nearly equivalent to cutting off the tube at a point near the position of the highest hole that is open. This assumption will be examined in the present note (here we will confine our attention to the cylindrical-bore clarinet family), which will discuss also, as an extension of certain remarks made in the earlier paper, the properties of a bore with many closed side holes. Methods will be given for the effective lengths of pipes with many open side holes, and several questions of intonation will be examined insofar as they relate to the nature of the side holes. This discussion will be confined to the phenomena that occur in the lowest two registers of the clarinet, since the acoustic behavior of a cross-fingered pipe and the action of the register key bring in some subtleties having to do with the reed mechanism. These matters require a separate discussion that is outside the scope of the present set of notes.

Pipes with many closed side holes

A woodwind instrument having all its finger holes closed constitutes a tapered transmission line with side branches, and its characteristic impedance[3] and wave velocity depend upon the size and spacing of the holes along it. Because of the nature of our musical scale, the spacing of these side branches is not uniform along the pipe, the distance between a given hole and its nearest neighbors being about 6% of the length of the bore from the reed to that particular hole.

Elementary consideration of the closed holes in paper A showed the existence of a necessary relation between the interhole spacing $2s$ and the hole diameter $2b$, for a bore of diameter $2a$ having a wall thickness t. This matter will now be looked at more rigorously, since this type of bore forms the basis for all later analysis. Assume for the moment that we are dealing with a uniform bore with holes of uniform spacing and size, so that standard transmission-line theory can be applied in an obvious way. If we are interested in sounds whose wavelength is very long compared with the interhole spacing and wall thickness, we find that the characteristic impedance and the wave velocity are given by the following expressions:

$$Z_c = (\rho c/\pi a^2)(1 + D_c)^{-1/2}$$

$$v_c = c(1 + D_c)^{-1/2}$$

where

$$D_c = (1/2)(b/a)^2(t/s)$$

It is worth noting that due to the symmetry of the cosine and the antisymmetry of the tangent functions, these approximations are good to the 3rd order in ($\omega s/c$) and $\omega t/c$).[4]

Transmission-line theory in its simplest form is applicable to lines in which both propagation velocity and characteristic impedance remain constant along the line, so that the foregoing result proves the applicability of such theory to clarinet bores, since their normal-mode frequency ratios also require constant velocity and impedance throughout. In order to achieve this with a cylindrical bore, the parameter D_c must be kept constant by adjusting the hole radius to grow as the square root of the distance from the reed end, as explained in paper A.

It turns out that the presence of closed holes in a clarinet lowers the wave velocity by a few percent and also increases the effective cross-sectional area of the bore by the same amount. If for simplicity we imagine a bore free of side holes in its upper part, and this is joined to a slightly smaller bore provided with side holes so that the characteristic impedance is constant throughout, while the velocity differs in the two parts, we find that the effective length of the combination is altered by an amount $\Delta\ell$ due to the lowered sound velocity:

$$\Delta\ell = \ell\left[\frac{r - [1 - (v_c/c)]}{1 - r[1 - (v_c/c)]}\right]$$

Here r is the fraction of the complete bore that is provided with closed side holes. In actual clarinets the bore above the top hole is not altered in the way I have indicated (at least not in the simple and abrupt way shown here), and the neglect of this leads to a further length correction that is to be added to the one given here. This additional correction will be discussed later on in these notes.

Before leaving the closed-hole bore system it is worthwhile to remark on the fact that the system behaves as a low-pass filter whose cutoff is up above the 20th harmonic of the musical note being played, so that an examination of a bore allows a flat statement to be made about the highest frequency present in the sound spectrum!

Pipes with many open side holes

The size and spacing of the finger holes of a woodwind are jointly determined by the twin necessities of having the correct effective bore in that part of the instrument where the holes are closed (as discussed already) and by the need for obtaining a chromatic scale by successively opening the holes. Very fortunately for analysis purposes, the interaction between these two requirements is simple enough to permit a separate study of each at the beginning, after which they may be combined in a synthesis that permits accurate calculation of the acoustic properties of the bore and its side holes.

It has been thirty years since Richardson first applied impedance theory to the calculation of normal modes of a pipe with side holes,[5] but his formulation was only tractable when applied to the case of one or two open holes in a pipe. It was basically a graphical method from which it was difficult to draw any general conclusions. Attempts to apply transmission-line theory to the open-hole system of a woodwind were stalled (in the writer's mind, at least) by the common knowledge that the equations governing a lumped-constant line with variable coefficients become hopelessly complicated, a complication that appears to be aggravated by the fact that the coefficients change greatly in the distance of one (free-space) wavelength. These considerations overlook a familiar fact known to musicians and designers alike: that the pitch and tone quality of a given played note is almost independent of the size and position of all the open holes beyond the first two or three. I do not intend that this remark be taken as a rigorously correct statement, but rather as a strong hint on how an approximate theory may be constructed. The validity of the statement may then be evaluated in the light of the successes and failures of this theory.

The independence of the natural frequency from the spacing of the lower open holes implies that a meaningful theory might be constructed upon an assumed "equivalent" transmission line with holes of uniform size that are spaced according to the dimensions and spacing of the *first two open holes* in the actual instrument. A study of the impedance and propagation properties of this equivalent line will permit calculation of the acoustical behavior of several of the lowest vibrational modes of a real woodwind when account is taken of the composite nature of the system. That is, the actual instrument bore is treated as being made of three parts: a) a smooth tube between mouthpiece and the first closed hole, b) a lumped-constant uniform line (low-pass) created by the sequence of closed holes, and c) a lumped-constant high-pass line formed by the remaining holes, which are all open. The calculation of the resonant frequencies must be made for the complete system, and they must be further modified by the nature of the excitory system.

In the limit of low frequencies, the characteristic impedance and propagation constant of a uniform pipe with radius a and wall thickness t pierced by a row of holes of radius b spaced $2s$ apart may be written as follows:

$$Z_o = j(\rho c/3a^2)\,(2t_e/s)^{1/2}\,(1 + D_o)^{1/2}\,(a/b)$$

$$\cosh(s\Gamma_e) = (1 + 2D_o)$$

where

$$D_o = (1/2)(b/a)^2\,(s/t_e) = D_c \times (t \cdot t_e/s^2)$$

t_e = *effective length of the open hole* $\cong t + b \times f(b/a)$ [*see* note 6]

For frequencies lower than about the 6th harmonic or so of the fundamental note, the characteristic impedance is seen to be pure imaginary, and it is essentially masslike, so that a pipe terminated by a section pierced with many uniformly spaced holes behaves very much like one terminated by a small aperture. This behavior will be discussed further at a later time in this outline. The hyperbolic cosine has a magnitude greater than unity in the low-frequency region that is of primary musical interest, and this shows that the open-hole system operates as a high-pass filter below cutoff, so that there is no *wave* propagation in this part of the bore. The pressure variations are therefore in phase all along the open-hole bore, but their amplitudes are damped exponentially with a linear attenuation coefficient:

$$\alpha_o = (1/2s)\cosh^{-1}(1 + 2D_o)$$

so that the pressure amplitude p_i at the i'th hole is greater than that at the $(i + 1)$st hole (a distance $2s$ below it on the bore) by the amount

$$p_i/p_{i+1} = \exp[\cosh^{-1}(1 + 2D_o)]$$

The smaller holes on a clarinet or oboe are so spaced and proportioned that the calculated pressure amplitude falls by about 0.75 from hole to hole in good agreement with experimental measurement. It will be shown later that the angular distribution of the radiated sound will be strongly affected by this attenuation. I should like to emphasize that not all of the sound output of a clarinet or other woodwind falls within the qualitative domain of validity of the long-wavelength approximation described above. The fundamental assumption here is that the lower open holes of a real instrument do not determine the effective impedance presented to the closed-hole part of the bore. Mathematically this is represented by the fact that the open-hole part acts as a nonpropagating line, so that "messages" concerning the nonuniformity of the line are not carried back to the main bore. In the tonal spectrum of a clarinet note, those components that lie above the cutoff frequency implied by the general impedance formula of the uniform open-hole line will propagate with very little reflection down the open-hole part of the bore, and they are therefore radiated freely from *all* the holes.

Length corrections for closed-hole effects

For many years it has been customary to give the vibration frequencies of a complex resonator in terms of the length of some closely related resonator. A point that is sometimes neglected in regard to these end corrections is that they are almost always frequency dependent, so that the higher modes of oscillation of the complex resonator do not in general coincide with the higher modes of a simple resonator chosen to match at the lowest mode.

Because woodwinds having closed and open bores constitute a class of systems in which the velocity of sound may vary with frequency, and because both conical and cylindrical bores are used, there can be considerable confusion as to whether the acoustical length used in a given calculation is based on the (variable) internal sound velocity or the sound velocity in the open air (c). To avoid this ambiguity, and for other reasons having to do with the ease of computation, the length correction ($\Delta\ell_k$) for a complex pipe system whose desired *lowest-mode* frequency is ω_k will be given as a fraction F_k of the free-space quarter wavelength ℓ_k of this frequency for a clarinet type of bore (and of the free-space half wavelength for flute and oboe bores). The mechanical length L_k desired for the actual bore is then given by

$$L_k = \ell_k(1 - F_k)$$

Often it is important to know the value of the correction at frequencies that are integral multiples of ω_k , so that we may define $F_k(m)$ as the value of F_k at the m'th harmonic of ω_k.

In general there are several end corrections to be considered simultaneously: the mouthpiece cavity produces one of these, as does the blown reed; in addition to these are the corrections due to open and closed side holes, as well as corrections from the irregularities in the bore. These several corrections are additive in first order, so that if each correction is labelled by a subscript i, the complete equation for the length L_k of a bore that is to play the note ω_k is shown by:

$$L_k = \ell_k(1 - \sum_i F_{ik})$$

There is one such equation for each note in the low register of the instrument, the subscript k giving the serial number of the note beginning with the lowest in the scale. Once the various F's have been found, the design of a woodwind is straightforward. All that is required is to drill the holes for all the notes at the proper distances L_k from the reference mark at the top of the bore, after which the mouthpiece is attached so that it also is in the proper position relative to the same mark.

Effect of small variations in the cross-sectional area

In a clarinet bore there are small variations from the idealized cylindrical shape caused not only by irregularly spaced or duplicated finger holes, but also from deliberate alterations introduced by the builder in an attempt to improve the tone, response, and tuning of his instruments. In the present context we shall consider the bore to be "uniform" in its shape if the effective cross section (as defined in paper A and implied by the impedance relation given in the first equation of this paper) remains constant. The slowing-down of waves in a pipe with closed side holes gives rise to a separate end correction, as already explained, so that here we will take the wave velocity within the bore to be simply c.

Let $S_o(x)$ represent the bore cross-sectional area at a distance x from the upper end of the idealized pipe under consideration, so that $S_o(\ell) = \pi a^2$ where a is the radius of the open end. Let the variations of an actual bore from this idealized cross section be represented by a perturbing area $S_p(x)$ such that $(S_p/S_o) \ll 1$ at all points in the bore. The frequency of oscillation of the n'th mode of such a perturbed system may be accurately calculated by Rayleigh's method, making use of the wave functions belonging to the unperturbed bore. Such a calculation may be used to show that the perturbed system vibrates so that its n'th mode is equal in frequency to that of an idealized bore whose length is increased by the amount $\Delta\ell$ given below:

$$\Delta\ell = +\int_o^\ell (S_p/S_o)\cos\frac{(2n-1)\pi x}{\ell}\,dx$$

Examination of this expression shows that an enlargement of the bore in the neighborhood of a pressure node of the standing wave always raises the vibrational frequency, while an enlargement near an antinode of pressure lowers the frequency. As a result, the effect of a given perturbation of the bore may result in a sharpening or a flattening of the played note, depending on the note being played and on the register that is in use. Examples of this will be given later.

Effect of duplicated, missing, or mis-sized closed holes

The presence of a small hole of volume $V_i = \pi b_i^2 t_i$ located at the position x_i along the bore of an otherwise regular instrument may be represented mathematically by a delta function perturbation to the area:

$$S_p(x) = V_i\,\delta(x - x_i)$$

This perturbation is to be used in the potential energy term of the Rayleigh expression, but not in the kinetic term, since physical considerations show that the small extent of the hole in the axial direction of the bore precludes the possibility of much longitudinal motion in the hole. The complete discussion of the implications of this remark would take up too much room and so will not appear in this summary of results. The resulting end correction, which may also be found by impedance methods, is

$$\Delta \ell = (V_i/S_a) \cos^2 [\frac{(2n - 1)\pi x_i}{2\ell}]$$

Many woodwinds have duplicate side holes for certain notes, placed there for the purpose of easing the technical problems of the musician. The right-hand cross key of a clarinet, which is used for playing the B♮ below middle C, is an example, as is the doubly duplicated hole for the D♯ immediately above middle C. These extra holes give rise to pitch changes of varying amounts for all the notes played with one or more holes closed below them, in an amount determined by the equation above. At first thought it would seem logical to put a small constriction into the bore at the location of these extra holes, or a projecting lug that forms a "negative volume" equal and opposite to that introduced by the extra closed hole. There is however no possible way to do this without constricting the *flow* of air in the neighborhood of the hole and its correcting lug, so that there is a kinetic energy term added by the lug to the Rayleigh integrals that does not have a counterpart arising from the hole. Briefly we may say that a projection into the bore, or a constriction, must have its effect calculated from the equation just above the heading "Effect of duplicated, . . . ," while the extra closed hole produces changes that are to be calculated from the second equation after this same heading. The first of the equations I have just referred to shows that a constriction can either raise or lower the pitch, while the second of them shows only the possibility of a flattening. As a result, there is no way in which to ream the bore or to choose a projection into it that will compensate exactly for a duplicated closed hole.

An extension of the preceding calculations may be used to find the alteration in tuning produced by a closed side hole that is displaced from its "proper" position in the regular sequence of holes, and moved down or up the bore a short distance δ , so that it is located at the position $(x_i + \delta)$. In order to preserve the tuning of the note played with this particular hole open, the hole must be enlarged above its normal size if δ is positive, and decreased if δ is negative. The amount by which it must be enlarged is determined by the open-hole properties of the instrument. If the hole is displaced an amount δ and its volume is altered by an amount (ΔV) from the normal position and size for a hole belonging at the position x_i , the length change produced is

$$\Delta \ell = (\Delta V/S_o)\cos^2\left[\frac{(2n - 1)\pi x_i}{2\ell}\right] - (2n - 1)\pi (V_i/S_o)(\delta/\ell)\sin\left[\frac{(2n - 1)\pi x_i}{\ell}\right]$$

The first term has to do with the change in hole size, and this always produces a flattening effect if ΔV is positive independent of x_i or the mode number n. Although the amount of this flattening is strongly dependent on x_i and n, the maximum amplitude of this alteration is independent of n. The second term deals with the displacement of the hole and may be of either sign, with a maximum amplitude that grows in importance linearly with n.

Further remarks on closed-hole effects

It is worth comment that for cylindrical instruments such as the clarinet, the *fractional* length correction $F = \Delta\ell/\ell$ due to each closed hole perturbation is proportional to the ratio of the perturbing volume V_p to that of the complete bore $V_b = S_o\ell$. The exact magnitude of this correction is equal to this ratio multiplied by a simple trigonometric function of the hole position. As a result, the maximum (positive or negative) fractional frequency change that can possibly be caused by a hole is given in hundredths of a semitone by the following expression:

$$\textbf{maximum pitch change} = 1.66\,(V_p/V_b)\cdot 10^3 \ \textbf{cents}$$

An examination of the approximations involved in all of the length corrections shows them to be of the same general order of accuracy. The pitch error in cents to be expected in the final corrected length ($\ell + \Delta\ell$) for a single one of the several corrections may be shown to have a size something of the order of (m^2 x 10^{-3}) cents for a $\Delta\ell/\ell$ correction of m cents. This error is completely negligible from a musical point of view, as will become obvious from the examples that follow.

While the error arising from a single correction is negligible, the question arises as to whether or not the superposition of many such corrections does not lead ultimately to errors that are serious in their practical consequences for a designer. A simple estimate of this net error may be based on the fact that the *signs* of the errors contributed by the various corrections are essentially random and not correlated with the sign of the correction itself. It is a well-known principle of statistical analysis that the resultant of such a set of random quantities is given by the square root of the sum of the squares of the quantities themselves. Thus, if the j'th correction for a particular note has the magnitude m_j, the resultant error arising from the complete set of corrections is approximately given by

$$\textbf{resultant error} \cong 10^{-3}(\Sigma_j\, m_j^4)^{+1/2} \ \textbf{cents}$$

Numerical examples of closed-hole effects

The formulas given so far for the calculation of length corrections are simple and easily applied, but it is worthwhile to give a few selected examples to clarify their nature and practical order of magnitude. These examples will be based on a simplified representative clarinet bore of 15-mm diameter, and all results will be given as the fractional correction $F = \Delta\ell/\ell$ expressed in cents.

1) Suppose that the diameter of the mouthpiece and barrel joint is increased in diameter by 1 mm above that of the rest of the bore, this enlargement being extended down to a point 10 cm from the closed end of a tube of overall length 40 cm (roughly equivalent to the low B in the clarinet chalumeau register). We will also consider the case of a tube similarly enlarged,

with an overall length of 20 cm corresponding to the highest note that can be played in the lowest register of a clarinet.

40-cm tube:	n = 1	flattened 50 cents	(fundamental mode)
	n = 2	flattened 16 cents	(third harmonic)
	n = 3	sharpened 10 cents	
20-cm tube:	n = 1	flattened 70 cents	
	n = 2	sharpened 24 cents	
	n = 3	flattened 14 cents	

Two important results are illustrated by these numbers: a) the magnitude of the correction fluctuates strangely with mode number and b) a short bore is much more strongly affected by a perturbation than is a long one, and its fluctuations are completely different.

2) Many times when playing in an ensemble the musician has to tune to the other instruments by pulling out the mouthpiece and/or barrel joint. Let us assume that the mouthpiece is pulled out from the barrel joint a distance of 3 mm so that the perturbing volume in the gap is about 0.6 cm^3 located at a distance of 7 cm from the closed end. The volume of the bore itself is about 70 cm^3 if its length is 40 cm.

n = 1	flattened 13.4 cents
n = 2	flattened 6.5 cents
n = 3	flattened 0.6 cents

If on the other hand the instrument is lengthened by pulling it apart at the junction of the barrel joint and the rest of the instrument, we find the following set of flattenings:

n = 1	flattened 12.2 cents
n = 2	flattened 2.1 cents
n = 3	flattened 2.1 cents

Once again it is apparent that the intonation of the instrument is completely upset by the relatively small perturbation, and once more we see that these perturbations are irregular throughout the scale, so that the musician has to learn to re-correct each note separately, treating it as a special case.

(3) The two extra side holes for the alternate fingering of the D♯ above middle C on a clarinet are each about 6 mm in diameter and drilled through a wall that is about 7 mm thick at a position 27.5 cm from the upper end of the instrument. If we assume that the total length of the bore is once again 40 cm, we find that the extra holes alter the tuning of this instrument as follows:

n = 1 flattened 2 cents
n = 2 flattened 9 cents
n = 3 flattened 9 cents

Length corrections for open-hole effects (general)

The effective length of a clarinetlike bore terminating in any sort of impedance at the lower end is equal to the physical length of pipe that is used plus a correction length $\Delta\ell$ of pipe whose input impedance at the point of attachment to the main bore is exactly equal to the actual terminating impedance. Thus, for a closed-hole bore with characteristic impedance of Z_c at the junction and a terminating impedance Z_t:

$$jZ_c \tan(\omega \Delta \ell / v_c) = Z_t$$

This may then be solved explicitly for the value of $\Delta\ell$ at any desired frequency:

$$\Delta \ell = v_c / \omega) \tan^{-1}(-jZ_t / Z_c)$$

In general, $\Delta\ell$ is a complicated function of the frequency, but this causes no difficulty in the design of an instrument, since the frequency is pre-assigned by the nature of the musical scale. Once again it is often useful to employ a fractional correction of $F = \Delta\ell/\ell$ in the manner described earlier.

Pipe with a single side hole

The case of a bore pierced by a single side hole at a given distance from the lower end was solved by Richardson, as has already been mentioned, and it will serve as a simple introductory problem in the present context. Here the terminating impedance appearing at the bottom of the closed-hole part of the bore consists of the parallel input impedances of two ducts, one of which is the hole itself, and the other, the length of bore below the open hole.

Let the hole be of radius b drilled a distance M_e from the lower end of the instrument ($M_e = M$ plus an open-end correction at the bottom of the bore). As before, the effective thickness of the wall is t_e, and the radius of the bore is a. If we assume for simplicity that the upper part of the bore does not carry any closed holes, the length correction may be shown to have the form:

$$\Delta \ell = (v_c / \omega) \tan^{-1} \left[\frac{\tan(\omega t_e / c) \tan(\omega M_e / c)}{(b/a)^2 \tan(\omega M_e / c) + \tan(\omega t_e / c)} \right]$$

This result is valid for any lengths M_e and t_e, and for any frequency as long as the wavelength is considerably larger than the transverse dimensions of the hole and bore, as is always the case in musical instruments.

When the length M_e is much less than the wavelength, as is the practical case if only the lowest hole is open, we may make use of the low-frequency limit of the above expression, which turns out to be independent of frequency, making it a simple correction of the sort dear to tradition:

$$\Delta \ell = t_e\left[(b/a)^2 + (t_e/M_e)\right]^{-1}$$

This result may be used as a means for finding where to drill the first hole on a cylindrical bore that is to play a chromatic scale. Let ℓ_o be the desired effective length of the complete bore, and ℓ_1 be that of the bore when the first hole is opened. Geometrically this requires that $\ell_o = \ell_1 + M_e$, while musically we must require that $1.06\ \ell_1 = \ell_o$ if the hole is to raise the pitch one semitone. These two requirements may be shown equivalent to

$$0.06\ \ell_o - M_e + \Delta \ell = 0$$

which may be solved together with the preceding equation to give a value for M_e as shown below:

$$M_e = (1/2)(0.06\ \ell_o)\left\{1 + \left[1 + 4(a/b)^2 (t_e/0.06\ \ell_o)\right]^{1/2}\right\}$$

An interesting feature of this equation is the complete freedom it gives the designer in the choice of hole size and wall thickness. Once these have been chosen from mechanical considerations and the requirements laid down by the acoustical properties of the closed-hole part of the bore, one has only to drill the hole a distance M up from the open end. The freedom implied here is not absolute, however, inasmuch as the calculation is based on a low-frequency approximation. No serious difficulty will be experienced with the failure of this approximation, however, if the value of M_e satisfies the very conservative inequality

$$(\pi M_e/6\ell_o)^2 \ll 1$$

This is almost automatically satisfied in practice, unless the side hole is positively minute.

While the analytical form of the solution for M_e on an instrument having a true bell at the lower end is different from that given above, the general approach to its calculation is exactly the same, and the qualitative results described here may be carried over to the more complex case almost unchanged.

Pipe with many open side holes

A problem of greater significance in the theory of woodwinds is that of a bore ending in a section provided with a number of open side holes. We have seen that, in the musically interesting case where the interhole spacing is small compared with the wavelength, a pipe with open holes along it has an input impedance that is pure imaginary and of a masslike character. This impedance Z_o serves as the terminating impedance on the upper part of the bore when we set about calculating the effective length correction. There is however a serious trap concealed within this statement. It is fundamental to the nature of the impedance equations for periodic lines that the lines begin and end with "half-sections." The equations given earlier for Z_c and Z_o are based upon the assumption that each of the two types of line begins at a point *midway between the holes*, so that the $\Delta\ell$ implied above is to be measured from a point lying midway between the first open hole and the last closed one! For practical purposes we are more interested in a quantity that tells us how far up the bore from ℓ_k to drill the k'th hole (the first open hole in the series). As a result we must subtract the distance s (half the interhole spacing) in order to obtain the length correction measured from the first open hole. If we take, for example, a bore unencumbered with closed holes in its upper part, the length correction becomes (without making a low-frequency approximation)

$$\Delta\ell = (v_c/\omega)\tan^{-1}\left\{\frac{(b/a)^2(1/2)\cot(\omega t_e/c)\tan(\omega s/c) + 1}{(b/a)^2(1/2)\cot(\omega t_e/c)\cot(\omega s/c) - 1}\right\}^{1/2} - s$$

Examination of this expression, and of the more general one that includes the effects of closed holes in the upper part of the bore, shows that the value of $\Delta\ell$ remains quite constant over a frequency range extending rather close to the critical frequency at which the lower bore begins to "conduct" waves, as signaled by the change of sign in the denominator of Z_o. As a result of this approximate constancy of $\Delta\ell$, the normal-mode frequency *ratios* of a pipe ending with a sequence of open holes of the sort used in musical instruments lie within a tenth of a semitone of those of a simple pipe of the same effective length. Here we have an explicit mathematical justification for the assumptions made in the earlier paper on the negligible effect of the open-hole series on the results of that paper. One must recall, however, that these remarks are only valid for frequencies below the cutoff of the open-hole line.[7]

The correctness of this formulation of the length correction problem may be verified by measurements of clarinets, or by especially constructed bores with a minimum of perturbing complication. If one measures a distance upward from the first open hole equal to one-quarter wavelength reduced by the calculated $\Delta\ell$ for a particular note on a clarinet, a point on the mouthpiece is reached that is the nominal effective closed upper end of the bore for that note. If this procedure is carried out for several notes in the chalumeau register, all the nominal upper ends lie within 2 or 3 mm of each other. If a detailed analysis is made of the various perturbations to the bore caused by the irregularities in the closed-hole part of the bore, the effective closed end proves to be the same for all notes within a half millimeter, and this discrepancy is at the limit of accuracy imposed on the calculation by the difficulty of getting accurate data concerning the bore and its holes.

The diagrams that accompany these notes [no diagrams were with the carbon copy--VB] show the details of an experiment carried out with steel tubes of accurately known dimensions in conjunction with a digital frequency counter. The resonant frequencies of a 10" tube cut off squarely at the end are compared with those of a longer piece of the same tubing that is provided with a set of side holes accurately laid out along its lower length. The holes are spaced out in 6% increments of length from the closed plug, and their diameter grows as the square root of their distance from the plug, so that they are very similar to the holes on a clarinet. There are no formulas for calculating the effective length of a side hole, so that this quantity was measured for the first open hole by means of a Helmholtz resonator technique, as shown. A view of the accuracy of the experiments and of the calculation may be obtained by comparing the calculated and measured values for $\Delta\ell$ and making an analysis of the errors involved. For the lowest mode of vibration, the corrections are found to be the following:

calculated $\Delta\ell$	=	1.270 inches
experimental $\Delta\ell$	=	1.255 inches
discrepancy	=	0.015 inches

Most of the discrepancy is explainable.

A preliminary assessment of the uncertainties brought in by experimental errors and by the inexact value for the velocity of sound (which enters the correction only to second order) leads to the conclusion that only about 0.005 inches of the discrepancy can be attributed to these errors, leaving a systematic error yet to be explained. However, the agreement is still close enough for the formulas to be considered verified, since we find a discrepancy of only a few thousandths of an inch out of a total length of 10 inches. This amounts to a fractional error of 0.0015, or 2.5 *cents*, which would be applicable to every note in the low-register scale *in a systematic way*, so that it is trivially easy to correct in the design of an instrument.

Due to the goodness of the agreement, all the measured dimensions of the tubes will be redetermined in the near future, so that a careful estimate of the discrepancy and its error may be obtained. It is anticipated that the published version of these notes will include the results of this closer study.

Notes

[1]This article was found among Benade's papers as a pale carbon copy on pebbled second-sheet paper. Typed by Benade and left in a state of first revision, it paraphrases the On woodwind instrument bores, *J. Acoust. Soc. Am.* 31(2):137-146 (1959) but with less mathematical detail and with some added practical applications specific to the clarinet. It has been retyped by Virginia Benade and checked for accuracy of transcription by Peter Hoekje.

[2]A. H. Benade, On woodwind instrument bores, *J. Acoust. Soc. Am.* 31(2):137-146 (1959).

[3][note added by P. Hoekje] Benade often used the term characteristic impedance to refer to the specific acoustic impedance, which equals [density ρ] x [wave speed c]/[cross-sectional area πa^2].

[4][note added by P. Hoekje] This is an approximation that assumes that the only effect of the closed holes is on the compliance and not on the inertance of the bore. Keefe and Nederveen later have described the inertance due to wave-front spreading in the closed holes.

[5]E. G. Richardson, *Technical Aspects of Sound* (Elsevier, New York, 1953), Vol 1, 488.

[6][note added by P. Hoekje] $f(b/a)$ represents a coefficient due to the open-end correction, and Benade is implying that this varies with hole size; however, he often used $f(b/a) = 1.5$.

[7]For higher frequencies the analysis becomes more complicated, but may be studied by means of the WKB approximation. A calculation of this case is in progress.